Published by Institute of Physics Publishing for SISSA

Received: *January 5, 2006*
Accepted: *January 17, 2006*
Published: *February 2, 2006*

Schwinger-Dyson equation for non-Lagrangian field theory

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Abstract: A method is proposed of constructing quantum correlators for a general gauge system whose classical equations of motion do not necessarily follow from the least action principle. The idea of the method is in assigning a certain BRST operator $\hat{\Omega}$ to any classical equations of motion, Lagrangian or not. The generating functional of Green's functions is defined by the equation $\hat{\Omega}Z(J) = 0$ that is reduced to the standard Schwinger-Dyson equation whenever the classical field equations are Lagrangian. The corresponding probability amplitude Ψ of a field φ is defined by the same equation $\hat{\Omega}\Psi(\varphi) = 0$ although in another representation. When the classical dynamics are Lagrangian, the solution for $\Psi(\varphi)$ is reduced to the Feynman amplitude $e^{\frac{i}{\hbar}S}$, while in the non-Lagrangian case this amplitude can be a more general distribution.

Keywords: BRST Symmetry, BRST Quantization, Topological Field Theories, Anomalies in Field and String Theories.

 http://jhep.sissa.it/archive/papers/jhep022006007/jhep022006007.pdf

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1. Introduction

In this paper we introduce a generalization of the Schwinger-Dyson equation for dynamical systems whose classical equations of motion are not required to follow from the least action principle. The generating functional of Green's functions, being a solution of the generalized Schwinger-Dyson equation, defines quantum correlators in non-Lagrangian field theory.

It has been recently shown [1] that any classical field theory in d dimensions, be it Lagrangian or not, can be converted into an equivalent Lagrangian topological field theory (TFT) in $d+1$ dimensions. Path integral quantizing this effective Lagrangian TFT, one gets correlators for the original non-Lagrangian theory in d dimensions. If the original field theory admits an action principle, the path integral for the enveloping $(d+1)$-dimensional TFT can be explicitly integrated out in the bulk with the result that reproduces the standard Batalin-Vilkovisky quantization receipt [2, 3] for the original Lagrangian gauge theory. Thus, at least one systematic method is known of constructing quantum correlators for general non-Lagrangian systems. In the present paper, we formulate a generalization of the Schwinger-Dyson equation for non-Lagrangian theories that defines Green's functions in terms of the original space, i.e., without recourse to embedding into a Lagrangian TFT in $d+1$ dimensions. Below in the introduction, we give some elementary explanations of the main idea behind our construction. More rigorous and detailed exposition is given in the subsequent sections.

Consider the dynamics of fields $\phi = \{\phi^i\}$ governed by the equations of motion[1]

$$T_a(\phi) = 0\,. \tag{1.1}$$

[1]Hereinafter we use De Witt's condensed notation [4], whereby the superindex "i" comprises both the local coordinates on a d-dimensional space-time manifold M and possible discrete indices labelling different components of ϕ. As usual, the repeated superindices, e.g. $\phi^i\psi_i$, imply summation over the discrete indices and integration over the space-time coordinates w.r.t. an appropriate measure on M. The partial derivatives $\partial_i = \partial/\partial\phi^i$ are understood as variational ones.

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These equations are not assumed to come from the least action principle, so the indices "i" and "a", labelling the fields and the equations, may belong to different sets completely unrelated to each other. For a Lagrangian theory, however, these indices must coincide because

$$T_i(\phi) = \frac{\partial S(\phi)}{\partial \phi^i}\,, \tag{1.2}$$

where S is an action functional. The standard Schwinger-Dyson equation [5] for the generating functional of Green's functions $Z(J)$ reads

$$\hat{\mathbb{T}}_i Z(J) = 0, \qquad \hat{\mathbb{T}}_i = -J_i + T_i(\phi)|_{\phi\mapsto i\hbar\frac{\partial}{\partial J}}\,. \tag{1.3}$$

To avoid further restrictions on $Z(J)$, the operators $\hat{\mathbb{T}}_i$ must commute with each other, which requires the equations of motion to be Lagrangian:

$$[\hat{\mathbb{T}}_i, \hat{\mathbb{T}}_j] = i\hbar(\partial_i T_j - \partial_j T_i)|_{\phi\mapsto i\hbar\frac{\partial}{\partial J}} = 0 \quad \Leftrightarrow \quad T_i = \partial_i S\,, \tag{1.4}$$

for some action S.

One can regard the Schwinger-Dyson operators $\hat{\mathbb{T}}_i$ as those resulting from the canonical quantization of the abelian first-class constraints

$$\mathbb{T}_i(\phi, J) = T_i(\phi) - J_i\,, \qquad \{\mathbb{T}_i, \mathbb{T}_j\} = 0\,. \tag{1.5}$$

Here we consider the sources J_i as the momenta canonically conjugate to the fields ϕ^i,

$$\{\phi^i, \phi^j\} = 0\,, \qquad \{\phi^i, J_j\} = \delta^i_j\,, \qquad \{J_i, J_j\} = 0\,, \tag{1.6}$$

and use the momentum representation to pass from functions to operators:

$$\hat{J}_i = J_i\cdot\,, \qquad \hat{\phi}^i = i\hbar\frac{\partial}{\partial J^i}\,. \tag{1.7}$$

Since the constraints (1.5) are explicitly solved w.r.t. the momenta J, they must be abelian whenever they are first class. So, the property of the equations of motion to be Lagrangian is equivalent to the property of the constraints (1.5) to be first class. One can also quantize the constraints (1.5) in the coordinate representation related to the momentum one by the Fourier transform:

$$\hat{\mathbb{T}}_i = T_i(\phi) + i\hbar\frac{\partial}{\partial\phi^i}\,. \tag{1.8}$$

Imposing this constraint operator on a "state" $\Psi(\phi)$ which is the Fourier transform of the generating functional $Z(J)$, yields

$$\left[T_i(\phi)\, +\, i\hbar\frac{\partial}{\partial\phi^i}\right]\Psi(\phi) = 0\,. \tag{1.9}$$

For Lagrangian equations of motion (1.2), the solution is given by the Feynman probability amplitude

$$\Psi(\phi) = (\text{const})e^{\frac{i}{\hbar}S(\phi)}\,. \tag{1.10}$$

Thus, from the viewpoint of the phase space (1.6) of fields and sources, the Schwinger-Dyson equation is nothing but imposing the quantum first-class constraints (1.5) on the probability amplitude. Given the generating functional or corresponding probability amplitude, the quantum average of an observable $\mathcal{O}(\phi)$ is given by the path integral

$$\langle\mathcal{O}\rangle = \int D\phi\, \mathcal{O}(\phi)\Psi(\phi) = \mathcal{O}\left(i\hbar\frac{\partial}{\partial J}\right) Z(J)|_{J=0}\,, \tag{1.11}$$

where $D\phi$ is an integration measure on the configuration space of fields.

Notice that the constraints (1.5) start with the classical equations of motion $T(\phi)$ which are always Poisson commuting w.r.t. (1.6), be the equations Lagrangian or not. It is a common practice to regard the equations of motion as constraints in the space of fields, see e.g. [3], so it might be interesting to impose the classical equations themselves on the probability amplitude without adding the momentum term J to T:

$$T(\phi)\Psi(\phi) = 0\,. \tag{1.12}$$

The solution is obvious,

$$\Psi(\phi) \sim \delta(T(\phi))\,. \tag{1.13}$$

The function $\Psi(\phi)$ is the classical probability amplitude in the sense that the quantum average of any observable $\mathcal{O}(\phi)$ would be proportional to its classical on-shell value when calculated with this amplitude:

$$\langle\mathcal{O}\rangle = (\mathrm{const})\int \mathcal{D}\phi\, \delta(T(\phi))\mathcal{O}(\phi) \sim \mathcal{O}(\phi_0)\,, \tag{1.14}$$

ϕ_0 being a solution to the classical field equations (1.1). The classical probability amplitude (1.13) was introduced and studied by Gozzi et al [6] for Hamiltonian equations of motion. It was also shown in [6] that (1.13) is a classical limit of Feynman's amplitude (1.10).

A twofold general conclusion can be derived from these simple observations. First, taking the classical limit for the Schwinger-Dyson equation means the omission of the momentum term in the first class constraints (1.5). To put this another way, quantizing means extending the classical equations of motion, viewed as constraint operators imposed on the probability amplitude, by appropriate momentum terms so that the constraints remain first class. The second simple lesson is that the classical equations of motion, being viewed as quantum constraints, are first class, be the original equations Lagrangian or not. Regarding these observations, the way seems straightforward of constructing the generalization of the Schwinger-Dyson equation for non-Lagrangian classical systems: The Schwinger-Dyson operators are to be constructed by extending the general equations (1.1) with momentum terms. These momentum terms are to be sought for from the condition that the momentum-extended constraints must be first class.

Proceeding from the heuristic arguments above, we can take the following ansatz for the ϕJ-symbols of the Schwinger-Dyson operators:

$$\mathbb{T}_a(\phi, J) = T_a(\phi) + V_a^i(\phi)J_i + O(J^2)\,. \tag{1.15}$$

These operators are defined as formal power series in momenta (sources) J with leading terms being the classical equations of motion. Requiring the Hamiltonian constraints $\mathrm{T}_a = 0$ to be first class, i.e.,

$$\{\mathbb{T}_a, \mathbb{T}_b\} = U_{ab}^c \mathbb{T}_c \,, \qquad U_{ab}^c(\phi, J) = C_{ab}^c(\phi) + O(J) \,, \tag{1.16}$$

one gets an infinite set of relations on the structure coefficients in the expansion of $\mathbb{T}_a(\phi, J)$ in J. In particular, examining the involution relations (1.16) to zero order in J we get

$$V_a^i \partial_i T_b - V_b^i \partial_i T_a = C_{ab}^c T_c \,. \tag{1.17}$$

The value $V_a^i(\phi)$ defined by this relation is called the *Lagrange anchor*. It has been first introduced in the work [1] with a quite different motivation. The Lagrange anchor is a key geometric ingredient for converting a non-Lagrangian field theory in d dimensions into an equivalent Lagrangian TFT in $d+1$ dimensions [1].

If the field equations (1.1) are Lagrangian, one can choose the Lagrange anchor to be $-\delta_j^i$. This choice results in the standard Schwinger-Dyson operators (1.3), (1.5) having abelian involution (1.4). For general equations of motion (1.1), the Lagrange anchor has to be field-dependent (that implies non-abelian involution (1.16), in general) and is not necessarily invertible. Existence of the invertible Lagrange anchor is equivalent to existence of a Lagrangian for the equations (1.1). Zero anchor ($V = 0$) is admissible for any equations of motion, Lagrangian or non-Lagrangian, as it obviously satisfies (1.17). The zero anchor, however, leads to a pure classical equation (1.12) for the probability amplitude, and no quantum fluctuations would arise in such theory. Any nonzero Lagrange anchor, invertible or not, allows one to construct the generating functional of Green functions describing nontrivial quantum fluctuations.

Given the Lagrange anchor, the requirement for the symbols of the Schwinger-Dyson operators (1.15) to be the first class constraints (1.16) will recursively define all the higher terms in (1.15) as we explain in the next section. Since the general Lagrange anchor results in the non-abelian algebra (1.16) of the Schwinger-Dyson operators, the naive imposing of these operators on the generating functional $Z(J)$ or probability amplitude $\Psi(\phi)$

$$\hat{\mathbb{T}}_a \Psi(\phi) = 0 \tag{1.18}$$

could be not a self-consistent procedure. As always with non-abelian constraints, the BFV-BRST formalism [7, 3] gives the most systematic method to handle all the consistency conditions of the constraint algebra. Instead of naively imposing constraint operators, the BFV-BRST method implies seeking for the probability amplitudes defined as cohomology classes of the nilpotent operator associated to the first class constraints. So, constructing the BRST operator for the Schwinger-Dyson constraints (1.15), (1.16), we will get the equation defining quantum correlators for the non-Lagrangian theory. Notice that for the Lagrangian field theory, it was known long ago [10, 3] that the Schwinger-Dyson equation can be reinterpreted as the Ward identity for an appropriately introduced BRST symmetry. This idea, as we will see, still allows the derivation of the Schwinger-Dyson equation for non-Lagrangian field theory by making use of the BRST symmetry related with the Langange anchor.

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The paper is organized as follows. In section 2, we recall some results from ref. [1]. In particular, we describe the classical BRST complex one can associate with any (non)-Lagrangian gauge system. We also explain how the classical observables are described as BRST cohomology of this complex. In section 3, this BRST complex is used to construct the probability amplitude on the configuration space of fields. We also show that the equation defining this amplitude is reduced to the standard Schwinger-Dyson equation whenever the theory is Lagrangian. Section 4 gives the path integral representation for the probability amplitude and the quantum averages of physical observables. In section 5 we compare the proposed quantization scheme for (non)-Lagrangian gauge theories with the standard BV-quantization based on Lagrangian formalism. In the Lagrangian case, our method reproduces the BV scheme although some basic properties of quantum theory can have a more general form whenever no action principle is admissible for the classical dynamics. In particular, in the general non-Lagrangian case, the classical BRST differential can be a non-inner derivation of the antibracket and the probability amplitude is not necessarily an exponential of the master action.

2. Equations of motion as phase-space constraints

Consider a collection of fields $\phi = \{\phi^i\}$ subject to (differential) equations of motion

$$T_a(\phi) = 0 \tag{2.1}$$

and an admissible set of boundary conditions. For the sake of simplicity, assume that the fields ϕ are bosons; fermion fields and equations of motion can be easy incorporated into the formalism by inserting appropriate sign factors in the subsequent formulas.

It is convenient to think of ϕ as a coordinate system on an (infinite-dimensional) manifold $\mathcal{M}$ of all field configurations with prescribed boundary conditions; upon this interpretation one can regard $T = \{T_a(\phi)\}$ as a section of some vector bundle $\mathcal{E} \to \mathcal{M}$ over the base $\mathcal{M}$. The set $\phi_0 = \{\phi_0^i\} \subset \mathcal{M}$ of all solutions to eqs. (2.1) coincides then with the zero locus of the section $T \in \Gamma(\mathcal{E})$. Under the standard regularity conditions [3], $\{\phi_0\} \subset \mathcal{M}$ is a smooth submanifold associated with an orbit of gauge symmetry transformations (see eq. (2.4) below); for non-gauge invariant systems this orbit consists of one point.

Now let $T^*\mathcal{M}$ be a cotangent bundle of the space $\mathcal{M}$, with $\bar{\phi} = \{\bar{\phi}_i\}$ being the fiber coordinates. Using the physical terminology, we will refer to $\bar{\phi}$ as the momenta conjugate to the "position coordinates" ϕ. The canonical Poisson brackets on $T^*\mathcal{M}$ read

$$\{\phi^i, \phi^j\} = 0\,, \quad \{\phi^i, \bar{\phi}_j\} = \delta^i_j\,, \quad \{\bar{\phi}_i, \bar{\phi}_j\} = 0\,. \tag{2.2}$$

We can regard the original equations of motion (2.1) as holonomic constraints on the phase space $T^*\mathcal{M}$. From this viewpoint, the linearly dependent equations of motion correspond to reducible Hamiltonian constraints with Nöther's identity generators $Z = \{Z^a_A(\phi)\}$ playing the role of null-vectors for the constraints,

$$Z^a_A T_a = 0\,. \tag{2.3}$$

It may also happen that eqs. (2.1) are gauge invariant (i.e., they do not specify a unique solution), in which case there exists a set of nontrivial gauge symmetry generators $R = \{R^i_\alpha\}$ such that

$$R^i_\alpha \partial_i T_a = U^b_{\alpha a} T_b \,, \tag{2.4}$$

for some structure functions $U^b_{\alpha a}(\phi)$. Then we can enlarge the set of holonomic constraints $T_a = 0$ by the constraints

$$R_\alpha = R^i_\alpha(\phi)\bar{\phi}_i \tag{2.5}$$

that are linear in momenta.

Assume that the Nöther identity generators and the gauge symmetry generators are complete and irreducible in the usual sense [3]. Then, in consequence of (2.4) and completeness of the generators R, the combined set of constraints $\Theta_I = (T_a, R_\alpha)$ is first class:

$$\{\Theta_I, \Theta_J\} = U^K_{IJ}\Theta_K \quad \Leftrightarrow \quad \begin{aligned} &\{T_a, T_b\} = 0\,, \\ &\{R_\alpha, T_a\} = U^b_{\alpha a} T_b\,, \\ &\{R_\alpha, R_\beta\} = W^\gamma_{\alpha\beta} R_\gamma + T_a E^{ai}_{\alpha\beta}\bar{\phi}_i\,, \end{aligned} \tag{2.6}$$

$W^\gamma_{\alpha\beta}(\phi)$, $E^{ai}_{\alpha\beta}(\phi)$ being some structure functions. In view of (2.3) the constraints Θ are reducible

$$\Xi^I_A \Theta_I = 0\,, \qquad \Xi^I_A = (Z^a_A, 0)\,. \tag{2.7}$$

According to the terminology of [1], rels. (2.3), (2.4) define a gauge theory of type (1,1). Notice that the generators of gauge symmetry R and generators of Nöther's identities Z can be completely independent from each other in the case of non-Lagrangian theory [1]. In particular, it is possible to have gauge invariant but linearly independent equations of motion, and conversely, a theory may have linearly dependent equations without gauge invariance. In these cases we speak about gauge theories of type (1,0) and (0,1), respectively. The theories of type (0,0) are those for which eqs. (2.1) are independent and have a unique solution. The general (n, m)-type gauge theory with $n > 1$ and/or $m > 1$ corresponds to the case of $(n-1)$-times reducible generators of gauge transformations and $(m-1)$-times reducible generators of the Nöther identities (see [1] for the precise definition).

It is easy to see that the "number" of independent first class constraints among the Θ's coincides with the "number" of fields ϕ. The same can be said in a more formal way: the coisotropic surface $\mathcal{L} \subset T^*\mathcal{M}$ defined by the first-class constraints (2.6) is a Lagrangian submanifold. Consider the action of the Hamiltonian system with constraints Θ:

$$S[\lambda, \phi, \bar{\phi}] = \int_{t_1}^{t_2} dt(\bar{\phi}_i\dot{\phi}^i - \lambda^I\Theta_I(\phi, \bar{\phi}))\,. \tag{2.8}$$

This action corresponds to a purely topological field theory having no physical evolution w.r.t. to the time [2] t. The model is invariant under the standard gauge transformations

[2]Notice, that the "time" t in (2.8) is an auxiliary $(d+1)$-st dimension, which is not related anyhow with the evolution parameter in the (differential) equations of motion (2.1) . The true physical time is among the d original dimensions.

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generated by the first class constraints (2.6) and their null-vectors (2.7):

$$\delta_\varepsilon \phi^i = \{\phi^i, \Theta_I\}\varepsilon^I\,, \qquad \delta_\varepsilon \bar{\phi}_i = \{\bar{\phi}_i, \Theta_I\}\varepsilon^I\,,$$
$$\delta_\varepsilon \lambda^I = \dot{\varepsilon}^I - \lambda^K U^I_{KJ}\varepsilon^J + \Xi^I_A \varepsilon^A\,. \tag{2.9}$$

Here $\varepsilon^I = (\varepsilon^a, \varepsilon^\alpha)$ and ε^A are infinitesimal gauge parameters, and the structure functions $U^I_{KJ}(\phi)$ are defined by (2.6).

Imposing the boundary conditions on the momenta

$$\bar{\phi}_i(t_1) = \bar{\phi}_i(t_2) = 0\,, \tag{2.10}$$

one can see that dynamics of the model (2.8) is equivalent to that described by the original equations (2.1). Indeed, let $\gamma(t) = (\phi(t), \bar{\phi}(t), \lambda(t))$ be a trajectory minimizing the action (2.8). Due to the equations of motion

$$\frac{\partial S}{\partial \lambda^I} = 0\,, \tag{2.11}$$

the phase-space projection $(\phi(t), \bar{\phi}(t))$ of $\gamma(t)$ lies on the constraint surface $\mathcal{L}$. Given a time moment $t_0 \in (t_1, t_2)$, one can always find an appropriate gauge transformation (2.9) moving the point $(\phi(t_0), \bar{\phi}(t_0)) \in \mathcal{L}$ to any other point of the constraint surface and simultaneously assigning any given value to $\lambda(t_0)$, no matter how the boundary points $\gamma(t_1), \gamma(t_2)$ were fixed. So, there are no true physical dynamics in the bulk (t_1, t_2) and the only nontrivial equations to solve are the constraints (2.11) on the boundary values of fields $(\phi(t), \bar{\phi}(t))$ at t_1 and t_2. But the boundary condition (2.10) reduces eqs. (2.11) to the original equations of motion $T_a(\phi(t_1)) = T_a(\phi(t_2)) = 0$. Thus, we have two copies of the original dynamics corresponding to the end points of the "time" interval $[t_1, t_2]$.

It is easy to see that the quantization of the topological model (2.8) induces a trivial quantization of the original theory [1]: the quantum averages of physical observables coincide exactly with their classical values, i.e., no quantum corrections arise. In a theory without gauge symmetries and identities, this is clearly seen from rels. (1.12)-(1.14). To get a nontrivial quantization, one has to modify the model (2.8) keeping intact its classical dynamics. For this end we consider a formal deformation of the constraints (2.6) and their null-vectors (2.7) by higher powers of momenta,

$$\tilde{\Theta}_I = (\tilde{T}_a, \tilde{R}_\alpha)\,, \qquad \begin{aligned} \tilde{T}_a &= T_a(\phi) + V^i_a(\phi)\bar{\phi}_i + O(\bar{\phi}^2)\,, \\ \tilde{R}_\alpha &= R^i_\alpha(\phi)\bar{\phi}_i + O(\bar{\phi}^2)\,, \\ \tilde{Z}^a_A &= Z^a_A(\phi) + O(\bar{\phi})\,. \end{aligned} \tag{2.12}$$

In order for the new constraints $\tilde{\Theta}_I$ to define a topological model - a theory without physical degrees of freedom in the bulk of the time interval - they must be first class and have $\tilde{\Xi}^I_A = (\tilde{Z}^a_A, 0)$ as null-vectors. These requirements result in an infinite set of relations on the expansion coefficients in (2.12). In particular, at the zeroth order in $\bar{\phi}$'s we get

$$V^i_a \partial_i T_b - V^i_b \partial_i T_a = C^c_{ab} T_c\,, \tag{2.13}$$

cf. (1.15-1.17). The condition (2.13) is necessary and sufficient for the existence of higher order deformations respecting the algebraic structure of constraints [1]. The functions (V_a^i, C_{ab}^c) are said to define a *Lagrange structure* compatible with the equations of motion (1.1); the set $V = \{V_a^i\}$, determining the first order deformation, is called the Lagrange anchor. Clearly, the condition (2.13) does not specify a unique Lagrange structure. As was shown in [1] any two Lagrange structures are related to each other by the following transformation:

$$V_a^i \to V_a^i + T_b G_a^{bi} + G_a^\alpha R_\alpha^i + \partial_j T_a G^{ij}\,, \qquad C_{ab}^c \to C_{ab}^c - G_{[a}^{ci}\partial_i T_{b]} - G_{[a}^\alpha U_{\alpha b]}^c + G_{ab}^A Z_A^c\,, \tag{2.14}$$

where G_{ab}^A, G_a^{bi}, G_a^α, $G^{ij} = G^{ji}$ are arbitrary functions and the square brackets mean antisymmetrization in indices a, b. In particular, one can use these transformations to generate a nontrivial Lagrange structure from the trivial one $V = C = 0$.

Now consider the Hamiltonian action (2.8) with the constraints $\tilde{\Theta}$ in place of Θ. We claim that this replacement does not affect the classical dynamics. Indeed, repeating the arguments above, we conclude that the classical dynamics are still localized at the boundary; but condition (2.10) reduces (2.11) to the original equations of motion (2.1) for any formal deformation (2.12). The classical equivalence, however, is not followed by the quantum one, and as we will see in the next section, it is the nontrivial Lagrange anchor V that defines possible "directions" of nontrivial quantum fluctuations near the classical solution.

The BRST-BFV quantization of the Hamiltonian constrained system implies the extension of the original phase space $T^*\mathcal{M}$ by ghost variables [7, 3]. To each first class constraint $\tilde{\Theta}_I = (\tilde{T}_a, \tilde{R}_\alpha)$ we assign the ghost field $\mathcal{C}^I = (\bar{\eta}^a, c^\alpha)$ and the conjugate ghost momentum $\bar{\mathcal{P}}_I = (\eta_a, \bar{c}_\alpha)$. Since the constraints $\tilde{\Theta}$ are supposed to be reducible, we also introduce the canonically conjugate pairs of ghosts-of-ghosts $(\bar{\xi}^A, \xi_A)$. The canonical Poisson structure on the ghost extended phase-space is defined as follows:

$$\{\mathcal{C}^I, \bar{\mathcal{P}}_J\} = -\delta_J^I\,, \quad \{\xi_B, \bar{\xi}^A\} = -\delta_B^A\,. \tag{2.15}$$

The Poisson brackets vanish among other variables with the exception of ones in the original phase space $T^*\mathcal{M}$, which are left unchanged[3]. The Grassman parity and the ghost number assignments of the new fields are given by

$$\begin{aligned} &\epsilon(\mathcal{C}^I) = \epsilon(\bar{\mathcal{P}}_I) = 1\,, \qquad &&\epsilon(\bar{\xi}^A) = \epsilon(\xi_A) = 0\,, \\ &\mathrm{gh}(\mathcal{C}^I) = -\mathrm{gh}(\bar{\mathcal{P}}_I) = 1\,, \quad &&\mathrm{gh}(\bar{\xi}^A) = -\mathrm{gh}(\xi_A) = 2\,. \end{aligned} \tag{2.16}$$

Upon the complex conjugation the phase-space variables behave as

$$\phi^* = \phi\,, \qquad \bar{\phi}^* = \bar{\phi}\,, \qquad \mathcal{C}^* = \mathcal{C}\,, \qquad \bar{\mathcal{P}}^* = -\bar{\mathcal{P}}\,, \qquad \xi^* = -\xi\,, \qquad \bar{\xi}^* = -\bar{\xi}\,. \tag{2.17}$$

[3]Geometrically, one can regard the ghost variables as linear coordinates in the fibers of a $\mathbb{Z} \times \mathbb{Z}_2$-graded vector bundle E over the configuration space $\mathcal{M}$. Upon this interpretation the canonical Poisson brackets on the extended phase space - the total space of E - are modified by terms involving a linear connection on E (see [11, 12, 1], for more detailed discussion). Here, however, we consider only the simplest case where $E \to \mathcal{M}$ is a trivial vector bundle with flat connection.

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The gauge structure of the topological model is completely encoded by the BRST charge

$$\Omega = \mathcal{C}^I \tilde{\Theta}_I + \bar{\xi}^A \tilde{\Xi}_A^I \bar{\mathcal{P}}_I + \cdots . \tag{2.18}$$

By definition,

$$\Omega^* = \Omega\,, \qquad \mathrm{gh}(\Omega) = 1\,, \qquad \epsilon(\Omega) = 1\,, \tag{2.19}$$

and the higher orders of ghost variables in (2.18) are determined from the master equation

$$\{\Omega, \Omega\} = 0\,. \tag{2.20}$$

Denote by $\mathcal{F}$ the Poisson algebra of functions on the ghost-extended phase space. The generic element of $\mathcal{F}$ is given by formal power series in $\mathcal{C}, \bar{\mathcal{P}}, \xi, \bar{\xi}$ and $\bar{\phi}$ with coefficients being smooth (in any suitable sense) functions of ϕ. The adjoint action of Ω makes $\mathcal{F}$ a cochain complex: A function F is said to be BRST-closed (or BRST-invariant) if $\{\Omega, F\} = 0$, and a function B is said to be BRST-exact (or trivial) if $B = \{\Omega, C\}$, for some C. The corresponding cohomology group $\mathcal{H} = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k$ is naturally graded by the ghost number.

As usual, the physical observables are nontrivial BRST invariants with ghost number zero. It can be shown [1] that the cohomology class of any BRST-invariant function $F = F(\phi, \bar{\phi}, \mathcal{C}, \bar{\mathcal{P}}, \xi, \bar{\xi})$ is completely determined by the projection of F on $\mathcal{M}$, i.e., by the function $\bar{F}(\phi) = F(\phi, 0, 0, 0, 0, 0)$, and a function $\mathcal{O}(\phi)$ is the projection of some physical observable iff

$$R_\alpha^i \partial_i \mathcal{O} = F_\alpha^b T_b \tag{2.21}$$

for some $F_\alpha^b(\phi)$. Thus, to any on-shell gauge-invariant function $\mathcal{O}$ on $\mathcal{M}$ one can associate a BRST-invariant function on the extended phase space and vice versa. Let $[F]$ denote the BRST-cohomology class of a physical observable F, then the map

$$[F] \mapsto \langle F \rangle_0 \equiv \bar{F}(\phi_0) \in \mathbb{R}\,, \tag{2.22}$$

with ϕ_0 being a unique solution to eqs. (2.1), establishes the isomorphism $\mathcal{H}_0 \simeq \mathbb{R}$. By definition, $\langle F \rangle_0$ is the classical (expectation) value of the physical observable F.

One can also use the BRST language to give another proof of the classical equivalence of the topological theories associated with the constraints Θ and $\tilde{\Theta}$ involving trivial and nontrivial Lagrange anchors, respectively. Let Ω_1 denote the BRST charge constructed by the former set of constraints. Then the BRST charge (2.18) is proved to be canonically equivalent to Ω_1. Namely, there exists a function $G \in \mathcal{F}$ of ghost number zero such that

$$\Omega = e^{\{G,\,\cdot\,\}} \Omega_1 = \sum_{k=0}^{\infty} \frac{1}{k!} \{G, \{G, \ldots \{G, \Omega_1\} \ldots\}\,. \tag{2.23}$$

The canonical equivalence of these two BRST charges implies the isomorphism between corresponding cohomology groups, and hence the physical equivalence of the classical theories defined by Ω and Ω_1. In terms of the original constrained dynamics on $T^*\mathcal{M}$, rel. (2.23) implies the absence of nontrivial deformations for the regular constraint systems Θ. In other words, any deformation (2.12) is obtained by a trivial superposition of a canonical transform of $T^*\mathcal{M}$ and a linear combining of the initial constraints, $\Theta_I \to G_I^J\Theta_J$, with some nondegenerate matrix $G_I^J(\phi,\bar{\phi})$.

Example 1. Let $S(\phi)$ be a nonsingular action functional, so that the corresponding Van-Vleck's matrix $S_{ij} \equiv \partial_i\partial_j S$ is invertible in a sufficiently small vicinity of a classical solution ϕ_0. The BRST charges corresponding to the zero and the canonical anchors read

$$\Omega_1 = \bar{\eta}^i\partial_i S\,, \qquad \Omega = \bar{\eta}^i(\partial_i S - \bar{\phi}_i)\,. \tag{2.24}$$

Let us show that the two BRST charges (2.24) are related to each other by the canonical transform (2.23) with G being a function of ϕ and $\bar{\phi}$. To this end, we first split the phase-space variables onto the "position coordinates" $\varphi^I = (\phi^i, \eta_i)$ and their conjugate momenta $\bar{\varphi}_J = (\bar{\phi}_j, \bar{\eta}^j)$, and introduce an auxiliary $\mathbb{N}$-grading counting the total number of momenta entering monomials in $\bar{\varphi}$'s (the m-degree in the terminology of ref. [1]). By definition, $\mathcal{F} = \bigoplus_{k\in\mathbb{N}}\mathcal{F}_k$, where $\mathcal{F}_k$ consists of homogeneous functions of m-degree k:

$$\mathcal{F}_k \ni F \;\Leftrightarrow\; NF = kF\,, \qquad N = \bar{\varphi}_I\frac{\partial}{\partial\bar{\varphi}_I}\,. \tag{2.25}$$

Now we can prove the statement above by induction on the m-degree. Observe that

$$\Omega^{(3)} \equiv e^{\{G_2,\,\cdot\,\}}\Omega_1 = \Omega + O(\bar{\varphi}^3)\,, \qquad G_2 = \frac{1}{2}S^{ij}\bar{\phi}_i\bar{\phi}_j\,, \tag{2.26}$$

S^{ij} being the matrix inverse to S_{ij}. In other words, the function $\Omega^{(3)}$ is canonically equivalent to Ω modulo $\bar{\varphi}^3$. Suppose $\Omega^{(k)}$ is canonically equivalent to Ω modulo k, i.e.,

$$\Omega^{(k)} = \Omega + \Omega_k + O(\bar{\varphi}^{k+1})\,, \qquad \Omega_k \in \mathcal{F}_k\,. \tag{2.27}$$

It then follows from the master equation $\{\Omega^{(k)},\Omega^{(k)}\} = 0$ that

$$\delta\Omega_k \equiv \{\Omega_1,\Omega_k\} = 0\,, \tag{2.28}$$

where we have introduced the nilpotent differential $\delta : \mathcal{F}_k \to \mathcal{F}_k$ preserving m-degree. Explicitly,

$$\delta = -\partial_i S\frac{\partial}{\partial\eta_i} + \bar{\eta}^i S_{ij}\frac{\partial}{\partial\bar{\phi}_j}\,, \qquad \delta^2 = 0\,. \tag{2.29}$$

Notice that the δ-cohomology is nested in $\mathcal{F}_0$, as we have the following contracting homotopy for N with respect to δ:

$$\delta^* = \bar{\phi}_i S^{ij}\frac{\partial}{\partial\bar{\eta}^j}\,, \qquad \delta\delta^* + \delta^*\delta = N\,. \tag{2.30}$$

Then for any $k > 0$ we have

$$\Omega_k = \delta G_k\,, \qquad G_k \equiv \frac{1}{k}\delta^*\Omega_k\,, \tag{2.31}$$

and hence

$$\Omega^{(k+1)} \equiv e^{\{G_k,\,\cdot\,\}}\Omega^{(k)} = \Omega + O(\bar{\varphi}^{k+1})\,. \tag{2.32}$$

Now we define the desired G through the limit

$$e^{\{G,\cdot\}} = \lim_{k\to\infty} e^{\{G_k,\cdot\}} \circ \cdots \circ e^{\{G_3,\cdot\}} \circ e^{\{G_2,\cdot\}} . \tag{2.33}$$

By construction, $e^{\{G,\cdot\}}\Omega_1 = \Omega$. We leave it to the reader to check that the generator

$$G = \frac{1}{2}S^{ij}\bar{\phi}_i\bar{\phi}_j - \frac{1}{12}S^{kl}\partial_l S^{ij}\bar{\phi}_i\bar{\phi}_j\bar{\phi}_k + O(\bar{\phi}^4) , \tag{2.34}$$

being defined by (2.26), (2.31) and (2.33), does not actually depend on η's and $\bar{\eta}$'s. It may be shown that the on-shell value of G, i.e., the function $W_{\mathrm{tree}}(\bar{\phi}) \equiv G(\phi_0, \bar{\phi})$, coincides with the generating function of connected Green's functions in tree approximation.

3. Probability amplitudes and a generalized Schwinger-Dyson equation

In the previous section, a general gauge theory, whose equations of motion (2.1) are not necessarily Lagrangian, has been equivalently reformulated as a constrained Hamiltonian system in the phase space of fields and sources. Also the classical BFV-BRST formalism has been constructed for this effective constrained system that contains all the data concerning the original equations of motion (2.1), their gauge symmetries (2.4), and Nöther identities (2.3). The physical observables of this effective constrained system have been shown to be in one-to-one correspondence with the on-shell gauge invariants of the original non-Lagrangian theory. Now we are going to perform the operator BFV-BRST quantization [7−9, 3] of this effective constrained system with a view to studying physical states. By construction, the physical states of this effective constrained system are the wave functions on the (ghost-extended) space of all trajectories. Therefore corresponding matrix elements of physical operators describe quantum averaging over trajectories (histories) in the configuration space of fields. In other words these matrix elements are to be understood as the transition amplitudes of the original non-Lagrangian theory. In standard Lagrangian field theory these average values are usually described by Feynman's path integral (1.11). Below we demonstrate how the above definition of the transition amplitudes (in terms of the matrix elements of the BFV-BRST quantized effective constrained dynamics) reproduces the standard definition (1.11) in the Lagrangian case. In the non-Lagrangian case, however, the corresponding probability amplitude $\Psi(\phi)$ cannot be brought to Feynman's form (1.10) although it still defines a consistent quantum dynamics.

To perform the operator quantization of the extended phase space (2.2), (2.15) in the Schrödinger representation one should first divide the phase-space variables into "coordinates" and "momenta". For our purposes it is convenient to choose them as

$$\varphi^I = (\phi^i, \eta_a, \xi_A, c^\alpha) , \qquad \bar{\varphi}_J = (\bar{\phi}_i, \bar{\eta}^a, \bar{\xi}^A, \bar{c}_\alpha) \tag{3.1}$$

By construction, the expansion of the classical BRST charge (2.18) in powers of momenta contains no zeroth-order term:

$$\Omega = \sum_{k=1}^{\infty} \Omega_k , \qquad \Omega_k = \Omega^{I_1\cdots I_k}(\varphi)\bar{\varphi}_{I_1}\cdots\bar{\varphi}_{I_k} . \tag{3.2}$$

On substituting this expansion into the classical master equation (2.20) we get a chain of equations

$$\{\Omega_1,\Omega_1\}=0\,,\quad \{\Omega_1,\Omega_2\}=0\,,\quad \{\Omega_2,\Omega_2\}=-2\{\Omega_1,\Omega_3\}\,,\quad \text{etc}\,. \tag{3.3}$$

As is seen, the term linear in momenta, Ω_1, is Poisson-nilpotent by itself. In fact, Ω_1 is nothing but the "bare" BRST charge associated to the initial constraints (2.6). According to (2.23), it is canonically equivalent to the total BRST charge (3.2). Thus the leading term of the BRST charge (3.2) carries all the information about the classical gauge system (2.1), (2.3), (2.4), with no reference to the Lagrange structure. The Lagrange anchor V enters Ω_2, the term quadratic in momenta. The second equation in (3.3), being expanded in ghost variables, reproduces the defining relation for the Lagrange structure (2.13). Notice that the sum $\Omega_1+\Omega_2$ includes all the ingredients of the Lagrange structure; the higher order terms in (3.2) are added to get a Poisson-nilpotent function on the extended phase space. It follows form rels. (3.3) that Ω_2 defines the so-called *weak antibracket* among the momentum-independent functions:

$$(A,B)\equiv\{A,\{B,\Omega_2\}\}\,,\qquad \forall A(\varphi)\,,B(\varphi)\,. \tag{3.4}$$

In view of the third relation in (3.3) this antibracket satisfies the Jacobi identity up to the homotopy associated with the classical BRST-differential $DA\equiv\{A,\Omega_1\}$ (hence the name). The second relation in (3.3) implies that D differentiates the antibracket by the Leibniz rule. When the antibracket is degenerate, the operator D may well be a non-inner derivation of the anti-Poisson algebra, i.e., $D\neq(W\,,\,\cdot)$ in general. For a more detailed discussion of the S_∞-structure behind this antibracket see [1].

Let us start quantizing this effective constrained system in the Schrödinger representation, where the space of quantum states is a complex Hilbert space of functions of φ's w.r.t. the hermitian inner product

$$\langle\Psi_1|\Psi_2\rangle=\int D\varphi\Psi_1^*(\varphi)\Psi_2(\varphi)\,. \tag{3.5}$$

The integration measure $D\varphi$ is given by the direct limit of Lebesgue's measure upon the lattice approximation of field configurations. The operators corresponding to the phase-space variables act on the states by the rule

$$\hat{\varphi}^I\Psi=\varphi^I\Psi\,,\qquad \hat{\bar{\varphi}}_I\Psi=-i\hbar\frac{\partial\Psi}{\partial\varphi^I}\,, \tag{3.6}$$

where all the derivatives act from the left. By definition,

$$[\hat{\bar{\varphi}}_I,\hat{\bar{\varphi}}_J]=0\,,\qquad [\hat{\bar{\varphi}}_I,\hat{\varphi}^J]=-i\hbar\delta_I^J\,,\qquad [\hat{\varphi}^I,\hat{\varphi}^J]=0\,. \tag{3.7}$$

The hermiticity properties of the operators $\hat{\varphi}$ and $\hat{\bar{\varphi}}$ follow directly from (2.17), (3.1). The map

$$F(\varphi,\bar{\varphi})=\sum_{k=0}^{\infty}F^{I_1\cdots I_k}(\varphi)\bar{\varphi}_{I_1}\cdots\bar{\varphi}_{I_k}\;\mapsto\;\hat{F}=\sum_{k=0}^{\infty}F^{I_1\cdots I_k}(\hat{\varphi})\hat{\bar{\varphi}}_{I_1}\cdots\hat{\bar{\varphi}}_{I_k} \tag{3.8}$$

assigns to any function $F \in \mathcal{F}$ a unique pseudo-differential operator (the $\hat{\varphi}\hat{\bar{\varphi}}$-ordering prescription is applied). Identifying the elements of $\mathcal{F}$ with $\varphi\bar{\varphi}$-symbols of operators, one can be free to add to them any terms proportional to positive powers of $\hbar$ without any impact on the corresponding classical limit.

A crucial step in the operator BFV-BRST quantization [7, 3] is to assign a nilpotent operator $\hat{\Omega}$ to the classical BRST charge (2.18). The $\varphi\bar{\varphi}$-symbol of the operator $\hat{\Omega}$ is to have the form

$$\Omega(\varphi, \bar{\varphi}, \hbar) = \sum_{k=0}^{\infty} \hbar^k \Omega^{(k)}(\varphi, \bar{\varphi}) \,, \tag{3.9}$$

where the leading term $\Omega^{(0)}$ is given by the classical BRST charge (2.18), and the higher orders in $\hbar$ are determined by the requirements of hermiticity and nilpotency [9]:

$$\hat{\Omega}^\dagger = \hat{\Omega} \,, \qquad \hat{\Omega}^2 = 0 \,. \tag{3.10}$$

It may well happen that no $\hat{\Omega}$ exists satisfying these equations. In that case the classical theory admits no self-consistent quantization (based on $\varphi\bar{\varphi}$-symbols of operators). This is just the phenomenon usually called the quantum anomaly. In what follows we assume our theory to be anomaly free so that both equations (3.10) hold true.

Furthermore, we assume that the $\varphi\bar{\varphi}$-symbol of the quantum BRST charge (3.9) satisfies condition

$$\Omega(\varphi, 0, \hbar) = 0 \,. \tag{3.11}$$

In the theory we consider, this condition appears to be crucial for existence of nontrivial BRST-invariant states, i.e., states that are annihilated by the quantum BRST charge, $\hat{\Omega}|\Psi\rangle = 0$. The trivial states are the BRST-invariant states of the form $\hat{\Omega}|\Lambda\rangle$. Taking the quotient of the space of BRST-invariant states by the subspace of trivial ones gives the BRST-state cohomology.

The full BRST algebra includes, in addition to the nilpotent BRST charge, the ghost-number operator:

$$\hat{\mathcal{G}} = \frac{i}{2\hbar} \sum_I \mathrm{gh}(\varphi_I)[\hat{\varphi}^I \hat{\bar{\varphi}}_I + (-1)^{\epsilon(\varphi^I)} \hat{\bar{\varphi}}_I \hat{\varphi}^I] \,, \qquad \hat{\mathcal{G}}^\dagger = -\hat{\mathcal{G}} \,, \tag{3.12}$$

so that $[\hat{\mathcal{G}}, \hat{F}] = \mathrm{gh}(\hat{F})\hat{F}$ for any homogeneous $\hat{F}$. In particular,

$$[\hat{\mathcal{G}}, \hat{\Omega}] = \hat{\Omega} \,. \tag{3.13}$$

Under certain assumptions [3] the linear space of states splits as a sum of eigenspaces of $\hat{\mathcal{G}}$ with definite real ghost number. The physical states are usually associated with the equivalence classes of BRST-invariant states at ghost number zero. A consistent consideration of physical states in the Schrödinger representation is known to require further enlargement of the extended phase space by the so-called nonminimal variables [3]. These do not actually change the physical content of the theory as one gauges them out by adding appropriate terms to the original BRST charge. The nonminimal sector just serves to bring the physical states to the ghost-number zero subspace where one can endow them with a well-defined inner product.

The standard nonminimal sector includes the Lagrange multipliers $\lambda^I = (\lambda^a, \bar{\lambda}^\alpha)$ to the first class constraints (2.6), anti-ghosts $\bar{\mathcal{C}}_I = (\bar{\rho}_a, \rho_\alpha)$, ghosts-of-ghosts $(\sigma_A, \upsilon_A, \beta^A, \gamma^A)$ as well as their conjugate momenta $\pi_I = (\bar{\lambda}_a, \lambda_\alpha)$, $\mathcal{P}^I = (\rho^a, \bar{\rho}^\alpha)$, and $(\bar{\sigma}^A, \bar{\upsilon}^A, \bar{\beta}_A, \bar{\gamma}_A)$. The ghost number assignments of the new variables are given by

$$\begin{aligned} &\mathrm{gh}(\lambda^I) = 0\,, \quad \mathrm{gh}(\bar{\mathcal{C}}^I) = -1\,, \quad \mathrm{gh}(\sigma_A) = -1\,, \\ &\mathrm{gh}(\upsilon_A) = 0\,, \quad \mathrm{gh}(\beta^A) = 1\,, \qquad \mathrm{gh}(\gamma^A) = 2\,. \end{aligned} \tag{3.14}$$

The conjugate variables have the same parities but the opposite ghost numbers; since all the constraints are bosonic, the Grassman parity of any variable equals its ghost number modulo 2. We also impose the reality conditions

$$\pi^* = \pi\,, \qquad \bar{\mathcal{P}}^* = \bar{\mathcal{P}}\,, \qquad \bar{\sigma}^* = \bar{\sigma}\,, \qquad \upsilon^* = \upsilon\,, \qquad \bar{\beta}^* = \bar{\beta}\,, \qquad \gamma^* = \gamma\,, \tag{3.15}$$

so that the canonically conjugate variables are either real or pure imaginary depending on their Grassman's parity.

From now on we will denote the classical BRST charge (2.18) defined in the minimal sector by $\Omega_{\min}$*, while* Ω *will stand for a total BRST charge.* The latter is given by

$$\Omega = \Omega_{\min} + \pi_I \mathcal{P}^I + \bar{\beta}_A \gamma^A + \upsilon_A \bar{\sigma}^A\,. \tag{3.16}$$

Clearly, "putting hats" on the r.h.s. of (3.16) yields a hermitian and nilpotent operator $\hat{\Omega}$ whenever $\hat{\Omega}_{\min}$ was so. To promote the Schrödinger representation (3.6) to the fully extended phase space we place the variables $(\lambda^a, \lambda_\alpha, \rho^a, \rho_\alpha, \sigma_A, \upsilon_A, \beta^A, \gamma^A)$ among the coordinates φ and consider the other nonminimal variables as momenta belonging to the set $\bar{\varphi}$.

The physical states are defined as usual in the BRST theory:

$$\hat{\Omega}|\Phi\rangle = 0\,, \qquad \hat{\mathcal{G}}|\Phi\rangle = 0\,. \tag{3.17}$$

As the BRST operator $\hat{\Omega}$ corresponds to the first class constraints, being the deformed equations of motion and gauge symmetry generators, eqs. (3.17) define a probability amplitude on the ghost-extended space of trajectories. It is the equation which is to be understood as Schwinger-Dyson equation for the general (i.e., not necessarily Lagrangian) gauge theories.

The gauge system at hands being a topological one (in the sense discussed in the previous section), it might be naively expected to have a 1-dimensional subspace of physical states spanned by a unique (up to equivalence) "vacuum" state $|\Phi\rangle$. This would be quite natural because the probability amplitude has to be a unique function on the space of trajectories with prescribed boundary conditions. Actually, this is not always the case in the BRST theory: The physical dynamics may have several copies in the BRST-state cohomology[4], and choosing one of them is equivalent to imposing extra conditions on the

[4]A particular manifestation of this phenomenon is known as *doubling*, [3].

physical states over and above (3.17) (see [13 − 17, 3]). For example, taking $|\Phi'\rangle$ to be annihilated by all the momenta,

$$\hat{\bar{\varphi}}_I|\Phi'\rangle = 0\,, \tag{3.18}$$

we get a BRST-invariant state on account of (3.11). The conditions (3.18) also ensure zero ghost number for $|\Phi'\rangle$. Thus, $|\Phi'\rangle$ is a physical state.

A less trivial example of a physical state can be obtained by imposing the following extra conditions:

$$\begin{aligned} &\hat{\eta}_a|\Phi\rangle = \hat{\bar{c}}_\alpha|\Phi\rangle = \hat{\xi}_A|\Phi\rangle = \hat{\bar{\lambda}}^\alpha|\Phi\rangle = \hat{\lambda}^a|\Phi\rangle = 0\,, \\ &\hat{\rho}^a|\Phi\rangle = \hat{\bar{\rho}}^\alpha|\Phi\rangle = \hat{\gamma}^A|\Phi\rangle = \hat{\sigma}_A|\Phi\rangle = \hat{\beta}^A|\Phi\rangle = \hat{\upsilon}_A|\Phi\rangle = 0\,, \end{aligned} \tag{3.19}$$

Unlike $|\Phi'\rangle$, the form of the state $|\Phi\rangle$ essentially depends on a particular structure of constraints. In the Schrödinger representation (3.6) we have

$$\Phi(\varphi) = \delta(\eta)\delta(\xi)\delta(\lambda^a)\delta(\rho^a)\delta(\gamma)\delta(\sigma)\delta(\beta)\delta(\upsilon)\Psi(\phi)\,, \qquad \Phi'(\varphi) = c \in \mathbb{C}\,, \tag{3.20}$$

where the “matter state” Ψ is annihilated by the quantum constraint operators

$$\hat{\Theta}_I\Psi = 0 \quad \Leftrightarrow \quad \begin{cases} (T_a(\phi) - i\hbar V_a^i(\phi)\partial_i + \cdots)\Psi(\phi) = 0\,, \\ (-i\hbar R_\alpha^i(\phi)\partial_i + \cdots)\Psi(\phi) = 0\,. \end{cases} \tag{3.21}$$

Of course, the $\phi\bar{\phi}$-symbols of the quantum constraints $\hat{\Theta}_I$ may differ from $\Theta_I(\phi, \bar{\phi})$ by $\hbar$-corrections defined by the quantum master equation (2.20). In the absence of gauge symmetry eqs. (3.21) reproduce the equation (1.18) for the (complex conjugate of) probability amplitude of a (non-)Lagrangian field theory we have come to in Introduction on heuristical grounds. In that case the states $|\Phi\rangle$ and $|\Phi'\rangle$ are just *dual* to each other [3]. Given the wave function $\Phi^*(\varphi)$ - the probability amplitude on the extended space of histories - we can calculate the quantum average of any physical observable.

The physical observables are given by the BRST-cohomology classes of hermitian operators at ghost number zero:

$$\begin{aligned} &\hat{\mathcal{O}}^\dagger = \hat{\mathcal{O}}\,, \qquad [\hat{\Omega}, \hat{\mathcal{O}}] = 0\,, \qquad [\hat{\mathcal{G}}, \hat{\mathcal{O}}] = 0\,, \\ &\hat{\mathcal{O}} \sim \hat{\mathcal{O}} + [\hat{\Omega}, \hat{B}]\,, \quad \mathrm{gh}(\hat{B}) = -1\,. \end{aligned} \tag{3.22}$$

If $\mathcal{O}(\varphi, \bar{\varphi}, \hbar)$ is the $\varphi\bar{\varphi}$-symbol of an observable $\hat{\mathcal{O}}$, then $\mathcal{O}(\varphi, \bar{\varphi}, 0)$ is a classical observable in the sense of definition (2.21). As with the BRST charge, not any classical observable can be generally promoted to the quantum level because of anomalies.

As is known, the inner product (3.5) becomes ill-defined when restricted onto the physical subspace, and a regularization is needed [16, 3]. The usual receipt of regularizing inner product of two physical states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ reads

$$\langle\Phi_1|\Phi_2\rangle_K = \langle\Phi_1|e^{\frac{i}{\hbar}[\hat{\Omega}, \hat{K}]}|\Phi_2\rangle\,. \tag{3.23}$$

Here K is an appropriate gauge-fixing fermion of ghost number -1. Formally, the expression (3.24) does not depend on the choice of K, since we fold the regulator $e^{[\hat{\Omega}, \hat{K}]}$ between the states annihilated by $\hat{\Omega}$. (More precisely, it depends only on the homotopy class of K in the variety of all admissible gauge-fixing fermions.)

Now we postulate the number

$$\langle \mathcal{O} \rangle = \frac{\langle \Phi | \hat{\mathcal{O}} | \Phi' \rangle_K}{\langle \Phi | \Phi' \rangle_K} \tag{3.24}$$

to be the *quantum average* of a physical observable $\mathcal{O}(\phi)$ associated with the classical system (2.1) and the Lagrange structure (2.13).

The reader may wonder why we sandwich the physical observable between different copies of a single physical state, rather than take the diagonal matrix elements $\langle \Phi | \hat{\mathcal{O}} | \Phi \rangle_K$ or $\langle \Phi' | \hat{\mathcal{O}} | \Phi' \rangle_K$ by analogy with the definition of expectation values in conventional quantum mechanics. The reasons are as follows:

(i) Upon normalization all the aforementioned expressions give the same value for the quantum average (see Proposition 2 of the next section). In other words, either copy of the physical state is eligible.

(ii) It should be realized that given a physical observable $\mathcal{O}$, expression (3.24) defines actually an infinite number of transition amplitudes associated with different initial and final states of the original gauge theory (2.1) even though it looks like a particular equal-time matrix element of the effective Hamiltonian theory in $d+1$ dimensions. The different initial and final states (w.r.t. the true physical time containing among d dimensions) enter implicitly through the boundary conditions for the field ϕ in perfect analogy to the usual Feynman's transition amplitudes.

(iii) The special convenience of the "asymmetric" definition (3.24) is that it provides a direct link with the conventional path-integral quantization in the case of Lagrangian equations of motion (see Example 2 below). The relevance of asymmetric inner products has been long known [13] for tracing various relationships between the BRST theory and the Dirac quantization.

Rel. (3.24) has also the following "cohomological" interpretation similar to the definition of expectation values of classical physical observables (2.22). To any physical observable $\hat{\mathcal{O}}$ we can associate the physical state $|O\rangle = \hat{\mathcal{O}}|\Phi'\rangle$. The BRST-state cohomology being essentially one-dimensional, the state $|O\rangle$ has to be proportional to $|\Phi'\rangle$ up to a BRST-trivial state[5]. So, we have

$$|O\rangle \equiv \hat{\mathcal{O}}|\Phi'\rangle = \langle \mathcal{O} \rangle |\Phi'\rangle + \hat{\Omega}|\Lambda\rangle\,, \tag{3.25}$$

for some $\langle \mathcal{O} \rangle \in \mathbb{C}$ and $|\Lambda\rangle$. Upon passing to the BRST-cohomology, $|\Phi'\rangle$ becomes an eigenstate for any physical observable $\hat{\mathcal{O}}$, and taking the inner product of (3.25) with the BRST-invariant state $|\Phi\rangle$, we just identify the eigenvalue $\langle \mathcal{O} \rangle$ with the quantum average (3.24) of $\hat{\mathcal{O}}$. Using the definition (3.25), we can rewrite (3.24) in the form

$$\langle \mathcal{O} \rangle = \frac{\langle \Phi | O \rangle_K}{\langle \Phi | 1 \rangle_K} = (\mathrm{const}) \int D\varphi O(\varphi) \Phi^*_K(\varphi)\,, \tag{3.26}$$

that enables us to interpret the gauge-fixed probability amplitude $\Phi^*_K(\varphi) = \langle \Phi | \varphi \rangle_K$ as a linear functional on the space of physical observables represented by the states $O(\varphi) = \langle \varphi | O \rangle$.

[5] In principle, it may be proportional to any copy of $|\Phi\rangle'$, say $|\Phi\rangle$, but a close look at $|O\rangle$ rules out this possibility.

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Example 2. Consider a Lagrangian gauge theory of rank 1, which is described by an action $S(\phi)$ and a set of gauge algebra generators $R_\alpha = R^i_\alpha \partial_i$ subject to the relations

$$R^i_\alpha \partial_i S = 0\,, \qquad [R_\alpha, R_\beta] = W^\gamma_{\alpha\beta} R_\gamma\,. \tag{3.27}$$

The total and minimal BRST charges are given respectively by

$$\Omega = \Omega_{\rm min} + \bar{\lambda}_i \rho^i + \lambda_\alpha \bar{\rho}^\alpha + \bar{\beta}^\alpha \gamma_\alpha + \upsilon_\alpha \bar{\sigma}^\alpha\,, \qquad \Omega_{\rm min} = \Omega_1 + \Omega_2\,, \tag{3.28}$$

where

$$\begin{aligned}
\Omega_1 &= \bar{\eta}^i \partial_i S + c^\alpha (R^i_\alpha \bar{\phi}_i - \eta_j \partial_i R^j_\alpha \bar{\eta}^i) + c^\alpha \left(\frac{1}{2} c^\beta W^\gamma_{\beta\alpha} \bar{c}_\gamma - \xi_\gamma W^\gamma_{\beta\alpha} \bar{\xi}^\beta \right) \\
&\quad + \frac{1}{2} c^\alpha c^\beta \partial_i W^\gamma_{\beta\alpha} \xi_\gamma \bar{\eta}^i - \eta_i R^i_\alpha \bar{\xi}^\alpha\,, \\
\Omega_2 &= \bar{\eta}^i \bar{\phi}_i + \bar{c}_\alpha \bar{\xi}^\alpha
\end{aligned} \tag{3.29}$$

are terms linear and quadratic in momenta (cf. (3.2)). In the context of Lagrangian gauge theories the truncated BRST charge Ω_1 was first introduced in [18]. Quantizing the classical BRST charge (3.28) by the rule (3.8), one gets an odd second-order differential operator

$$\hat{\Omega} = -i\hbar \left(\partial_j S(\phi) - i\hbar \frac{\partial}{\partial \phi^j} \right) \frac{\partial}{\partial \eta_j} - i\hbar c^\alpha R^j_\alpha(\phi) \frac{\partial}{\partial \phi^j} + \cdots \tag{3.30}$$

acting on the Hilbert space of functions of

$$\varphi^I = (\phi^i, \eta_i, c^\alpha, \xi_\alpha, \lambda^i, \lambda_\alpha, \rho^i, \rho_\alpha, \sigma_\alpha, \upsilon_\alpha, \beta^\alpha, \gamma^\alpha)\,.$$

In general, this operator is neither hermitian nor nilpotent:

$$\hat{\Omega}^\dagger = \hat{\Omega} - 2i\hbar \hat{c}^\alpha \hat{A}_\alpha\,, \qquad \hat{\Omega}^2 = \hbar^2 (\hat{A}_\alpha \hat{\bar{\xi}}^\alpha + \widehat{\partial_i A} \hat{\bar{\eta}}^i)\,, \tag{3.31}$$

here the function

$$A \equiv c^\alpha A_\alpha = c^\alpha (\partial_i R^i_\alpha + W^\beta_{\alpha\beta}) \tag{3.32}$$

represents the so-called *modular class* of gauge algebra [11, 3]. Taking the hermitian part of $\hat{\Omega}$, we get a nilpotent BRST operator

$$\hat{\Omega}' = \frac{1}{2}(\hat{\Omega} + \hat{\Omega}^\dagger) = \hat{\Omega} - i\hbar \hat{A}\,, \qquad \hat{\Omega}'\hat{\Omega}' = 0\,. \tag{3.33}$$

Occurrence of the second term in $\hat{\Omega}'$ conflicts, however, with (3.11). As the result there may be no physical states unless the modular class of A is nontrivial. By construction, $\{\Omega_1, A\} = 0$, while the triviality means a stronger condition $A = \{\Omega_1, B\}$ for some $B(\varphi)$. In the latter case one can pass to an equivalent quantization by twisting the integration measure and the operator algebra,

$$D'\varphi = D\varphi e^{2B}\,, \qquad \hat{\varphi}'^I = e^{-B} \hat{\varphi}^I e^B = \hat{\varphi}^I\,, \qquad \hat{\bar{\varphi}}'_I = e^{-B} \hat{\bar{\varphi}}_I e^B = -i\hbar \partial_I - i\hbar \partial_I B\,, \tag{3.34}$$

so that the $\varphi'\bar{\varphi}'$-symbol of the operator (3.33) will satisfy (3.11) modulo $\hbar^2$. The nontrivial modular class indicates the presence of a genuine 1-loop anomaly. It should be noted that in local field theory the expressions like A are singular and proportional to $\delta(0)$, $\delta'(0)$, etc. It is beyond the scope of this paper to go into details of regularizing quantum divergences, so we restrict our consideration to the case $A = 0$. Then the solution to the generalized Schwinger-Dyson equation (3.17) reads

$$\Phi(\varphi) = (\mathrm{const})\, \delta(\eta)\delta(\xi)\delta(\rho^i)\delta(\lambda^i)\delta(\gamma)\delta(\beta)\delta(\sigma)\delta(\upsilon)\, e^{-\frac{i}{\hbar}S(\phi)} . \tag{3.35}$$

Now let $\mathcal{O}(\phi)$ be a classical observable of the original gauge theory (3.27), then

$$R^i_\alpha \partial_i \mathcal{O} = F^i_\alpha \partial_i S . \tag{3.36}$$

The wave function corresponding to the state $|O\rangle = \hat{O}|\Phi'\rangle$ has the form

$$O(\varphi) = \mathcal{O}(\phi) + c^\alpha F^i_\alpha(\phi)\eta_i + \frac{1}{2}c^\beta c^\alpha K^\gamma_{\alpha\beta}(\phi)\xi_\gamma + \cdots , \tag{3.37}$$

where dots stand for higher powers of η's and ξ's. Verifying the BRST-invariance of $|O\rangle$, we find

$$\hat{\Omega} O = \hbar^2 \Lambda , \qquad \Lambda = c^\alpha(\partial_i F^i_\alpha - K^\beta_{\alpha\beta}) + \cdots . \tag{3.38}$$

By definition, Λ is a BRST-closed state with ghost number 1. Moreover, it is separately annihilated by the $\hat{\Omega}_1$- and $\hat{\Omega}_2$-parts of the total BRST charge (3.28). If Λ is $\hat{\Omega}_1$-exact, i.e., $\hbar\Lambda = \hat{\Omega}_1 O^{(1)}$, we can cancel the r.h.s. of (3.38) out by redefinition $O' = O - \hbar O^{(1)}$, so that the new state O' will be BRST-invariant up to $\hbar^2$; otherwise no quantum observable corresponds to the classical observable $\mathcal{O}$. Repeating this procedure time and again we can move the anomaly at higher orders in $\hbar$ until we are faced with a nontrivial $\hat{\Omega}_1$-cocycle at ghost number 1. In that case the procedure stops and we get an unavoidable (or genuine) quantum anomaly. Here we simply assume that $\Lambda = 0$.

To regularize the inner product in the physical subspace spanned by the state (3.35) and $\Phi' = 1$ let us take the gauge-fixing fermion

$$K = \frac{i}{\hbar}\rho_\alpha \chi^\alpha , \tag{3.39}$$

where the functions $\chi^\alpha(\phi)$ are chosen in such a way that the matrix $R^i_\alpha \partial_i \chi^\beta$ is nondegenerate. Then we have

$$[\hat{\Omega}, \hat{K}] = c^\alpha R^i_\alpha \partial_i \chi^\beta \rho_\beta + \lambda_\beta \chi^\beta - i\hbar \partial_j \chi^\beta \rho_\beta \frac{\partial}{\partial \eta_j} . \tag{3.40}$$

The gauge-fixed probability amplitude is given by

$$\Phi^*_K(\varphi) = e^{-\frac{i}{\hbar}[\hat{\Omega},\hat{K}]}\Phi^* = \delta(\eta_i - \partial_i\chi^\beta\rho_\beta)\delta(\xi)\delta(\lambda^a)\delta(\gamma)\delta(\beta)\delta(\sigma)\delta(\upsilon)e^{\frac{i}{\hbar}(S - c^\alpha R^i_\alpha \partial_i \chi^\beta \rho_\beta - \lambda_\beta \chi^\beta)} \tag{3.41}$$

Inserting (3.37) and (3.41) into the general expression (3.26) and integrating over arguments of δ-functions, we finally arrive at the Faddeev-Popov path-integral

$$\langle \mathcal{O} \rangle = (\mathrm{const}) \int D\phi Dc D\rho D\lambda\, \bar{O} \exp \frac{i}{\hbar}(S - c^\alpha R^i_\alpha \partial_i \chi^\beta \rho_\beta - \lambda_\alpha \chi^\alpha) , \tag{3.42}$$

where

$$\bar{O}(\phi, c, \rho) \equiv O(\phi, \eta, c, \xi)|_{\xi=0,\, \eta_i = \partial_i \chi^\beta \rho_\beta} = \mathcal{O}(\phi) + c^\alpha F(\phi)^i_\alpha \partial_i \chi^\beta \rho_\beta + \cdots , \tag{3.43}$$

and the normalization factor (const) is fixed by the condition $\langle 1 \rangle = 1$.

4. Path-integral representation for the probability amplitudes

We have seen that the general solution to the generalized Schwinger-Dyson equation (3.17) and auxiliary conditions (3.19) is given by the product of δ-functions of ghosts and Lagrange multipliers and the matter state $\Psi(\phi)$ defined by the constraint equations (3.21). The function $\Psi^*(\phi)$ is then identified with the probability amplitude on the original configuration space $\mathcal{M}$ of fields with given boundary conditions. The gauge invariant equations of motion give rise to the gauge invariant probability amplitude $\Psi^*(\phi)$, much as a gauge invariant action functional $S(\phi)$ leads to the gauge invariant Feynman's amplitude $e^{\frac{i}{\hbar}S}$ in the Lagrangian field theory. In both the cases an appropriate gauge-fixing procedure is required to compute the quantum averages of physical observables. The crucial difference, however, is that in the conventional BV quantization the probability amplitude is known from the outset, whereas in non-Lagrangian theory this amplitude is yet to be found from the constraint equation (3.21), which is hard to solve in general. A perturbation solution can be obtained by making use of a suitable path-integral representation that we are proceeding to derive.

Let us start with the following two propositions.

Proposition 1. Let $|\phi\rangle$ be a family of physical states defined by the conditions

$$\begin{gathered}\hat{\Gamma}|\phi\rangle = 0\,, \qquad \hat{\phi}^i|\phi\rangle = \phi^i|\phi\rangle\,, \\ \Gamma = (\mathcal{C}^I, \bar{\xi}^A, \pi_I, \bar{\mathcal{C}}_I, \bar{\sigma}^A, \bar{\upsilon}^A, \bar{\beta}_A, \bar{\gamma}_A)\,,\end{gathered} \tag{4.1}$$

and $\Psi(\phi)$ be a solution of the constraint equation (3.21), then

$$\Psi(\phi_1)\Psi^*(\phi_0) = \langle\phi_1|\phi_0\rangle_K\,, \tag{4.2}$$

for some admissible gauge-fixing fermion K.

The proof can be found in [17]. Notice that the states $|\phi\rangle$ are physically equivalent to each other and have well-defined inner product with the probability amplitude (3.20), (3.21) on the extended configuration space,

$$\langle\phi|\Phi\rangle = \Psi(\phi)\,. \tag{4.3}$$

The last property characterizes the state $|\phi\rangle$ as *dual* to $|\Phi\rangle$.

Proposition 2. The quantum average (3.24) of a physical observable $\mathcal{O}$ can be written as

$$\langle\mathcal{O}\rangle = \frac{\langle\Phi'|\hat{\mathcal{O}}|\Phi'\rangle_K}{\langle\Phi'|\Phi'\rangle_K}\,, \tag{4.4}$$

where $|\Phi'\rangle$ is the physical state (3.18) annihilated by all the momenta $\hat{\bar{\varphi}}$'s and K is an admissible gauge-fixing fermion.

Notice that unlike (3.24) the alternative definition (4.4) involves a single state $|\Phi'\rangle$ whose form is known and does not depend on a particular structure of constraints.

Sketch of the proof. The proof is based on abelianization arguments. Locally, one can always find a canonical transformation (2.23) bringing the total BRST charge Ω into the form

$$\Omega_1 = \Omega^I(\varphi)\bar{\varphi}_I = \bar{\eta}^a T_a + \bar{\xi}^A Z^a_A \eta_a + c^\alpha R^i_\alpha \bar{\phi}_i + \bar{\lambda}_a \rho^a + \lambda_\alpha \bar{\rho}^\alpha + \bar{\beta}_A \gamma^A + \upsilon_A \bar{\sigma}^A \,, \tag{4.5}$$

where the constraints T's and R's are in abelian involution,

$$R^i_\alpha \partial_i T_a = 0 \,, \qquad [R_\alpha, R_\beta] = 0 \,, \tag{4.6}$$

and $R^i_\alpha \partial_i Z^a_A = 0$. This canonical transform is then lifted to a unitary operator $\hat{U}$ such that $\hat{\Omega}_1 = \hat{U}\hat{\Omega}\hat{U}^\dagger$ and $\hat{U}|\Phi'\rangle = |\Phi'\rangle$. (See [8, 9] for the proof of quantum abelization theorem). Of course, the structure functions T's, Z's and R's (entering (4.5)) may differ from the original ones by quantum corrections that can be obtained from (3.10).

For abelian constraints we can take $\mathcal{O}$ to be strongly annihilated by the gauge symmetry generators,

$$R^i_\alpha \partial_i \mathcal{O} = 0 \,. \tag{4.7}$$

The function $\mathcal{O}(\phi)$ is thus BRST-invariant and does not depend on the ghost variables and Lagrange multipliers. Now consider the following gauge-fixing fermion:

$$K = \frac{i}{\hbar}[\eta_a \lambda^a + \rho_\alpha \chi^\alpha + \xi_A \beta^A + \bar{\gamma}_A N^A_a \bar{\eta}^a + \lambda^a N^A_a \sigma_A] \,, \qquad \det(Z^a_B N^A_a) \neq 0 \,. \tag{4.8}$$

We have $[\hat{\Omega}_1, \hat{K}] = i\hbar\widehat{\{\Omega_1, K\}}$, where

$$\begin{aligned} i\hbar\{\Omega_1, K\} = {} & (\lambda^a T_a + \rho^a \eta_a) + (\lambda_\alpha \chi^\alpha + c^\alpha R^i_\alpha \partial_i \chi^\beta \rho_\beta) + (\rho^a N^A_a \sigma_A + \lambda^a N^A_a \upsilon_A) \\ & + (\eta_a Z^a_A \beta^A + \xi_A \gamma^A) + (\bar{\gamma}_A N^A_a Z^a_B \bar{\xi}^B + \bar{\beta}_A N^A_a \bar{\eta}^a) \,. \end{aligned} \tag{4.9}$$

The last sum contains noncommuting terms that somewhat complicates calculations. Nonetheless, after some algebra one can find an explicit expression for the state $e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]}|\Phi'\rangle$. (We refer the reader to [16], where similar operatorial expressions were thoroughly studied as well as more general gauges.) For example, choosing the matrix N^A_a in such a way that $N^A_a Z^a_B = \delta^A_B$, we find

$$e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]}\Phi'(\varphi) = e^{\frac{i}{\hbar}[(\lambda^a T_a + \rho^a \eta_a) + (\lambda_\alpha \chi^\alpha + c^\alpha R^i_\alpha \partial_i \chi^\beta \rho_\beta) + (\rho^a N^A_a \sigma_A + \lambda^a N^A_a \upsilon_A) + c(\eta_a Z^a_A \beta^A + \xi_A \gamma^A)]} \tag{4.10}$$

for some nonzero constant $c \in \mathbb{R}$. Substituting (4.10) into (4.4) and integrating w.r.t. λ^a, λ_α, ξ_A, γ^A, c^α, ρ_α, and β^A, we get

$$\begin{aligned} \langle \Phi' | \hat{\mathcal{O}} e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]} | \Phi' \rangle = {} & (\text{const}) \int D\phi D\eta D\upsilon D\sigma \, \mathcal{O}(\phi) \delta(T_a + N^A_a \upsilon_A) \\ & \times \delta(Z^a_A \eta_a) \delta(\eta_a + N^A_a \sigma_A) \delta(\chi) \det(R^i_\alpha \partial_i \chi^\beta) \,. \end{aligned} \tag{4.11}$$

Further integration of υ, σ, and η yields

$$\langle \mathcal{O} \rangle = (\text{const}) \int D\phi \, \mathcal{O}(\phi) \Psi(\phi) \mu(\phi) \,, \tag{4.12}$$

where

$$\Psi = \delta(T_{\bar{a}}(\phi))\mu_0(\phi)\,, \qquad \mu_0 = \det(\delta^a_{\bar{a}}, Z^a_A)\,, \qquad \mu = \delta(\chi)\det(R^i_\alpha \partial_i \chi^\beta)\,, \tag{4.13}$$

$\{T_{\bar{a}}\} \subset \{T_a\}$ being a complete subset of independent constraints. The overall constant in (4.12) is determined by the normalization condition $\langle 1 \rangle = 1$.

Clearly, the distribution Ψ is nothing but the classical probability amplitude annihilated by the operators of abelian constraints (4.6). Substituting this Ψ into (3.20) gives a solution to the generalized Schwinger-Dyson equation (3.17) with $\hat{\Omega}_1$ in place of $\hat{\Omega}$. Explicitly,

$$\Phi(\varphi) = \delta(\eta)\delta(\xi)\delta(\lambda^a)\delta(\rho^a)\delta(\gamma)\delta(\sigma)\delta(\beta)\delta(\upsilon)\mu_0(\phi)\delta(T_{\bar{a}})\,. \tag{4.14}$$

The proof is completed by showing that the same expression (4.12) for the quantum average is obtained if one starts with the definition (3.24), where Ω and Φ are given respectively by (4.5) and (4.14), and chooses

$$K = \frac{i}{\hbar}\chi^\beta \rho_\beta \tag{4.15}$$

as gauge-fixing fermion. The details are left to the reader. $\square$

Regarding now the regulator $e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]}$, which is involved in the definition of inner products (3.23), as the evolution operator associated to the BRST-trivial Hamiltonian $[\hat{\Omega}, \hat{K}]$, one can immediately get the path-integral representations for expressions (4.2) and (4.4). For the probability amplitude we have

$$\Psi(\phi_1)\Psi^*(\phi_0) = \int \mathcal{D}\varphi\mathcal{D}\bar{\varphi} \exp \frac{i}{\hbar}\int_0^1 dt(\bar{\varphi}_I \dot{\varphi}^I - \{\Omega, K\})\,, \tag{4.16}$$

where, in accordance with (4.1), integration extends over all the paths obeying

$$\Gamma(0) = \Gamma(1) = 0\,, \qquad \phi^i(0) = \phi^i_0\,, \qquad \phi^i(1) = \phi^i_1\,. \tag{4.17}$$

The other variables are unrestricted at the endpoints. Integrating the function (4.16) w.r.t. ϕ_0 with an appropriate weight we get a desired solution of the constraint equations (3.21) as function of ϕ_1.

Similarly, the path integral

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\, O(\varphi(1)) \exp \frac{i}{\hbar}\int_0^1 dt(\bar{\varphi}_I \dot{\varphi}^I - \{\Omega, K\}) \tag{4.18}$$

with the boundary conditions

$$\bar{\varphi}_I(0) = \bar{\varphi}_I(1) = 0 \tag{4.19}$$

gives the quantum average of a physical observable $\mathcal{O}$. Here the value

$$O(\varphi) = \mathcal{O}(\phi) + (ghost\ terms) \tag{4.20}$$

is the $\bar{\varphi}$-independent part of a BRST-invariant extension of $\mathcal{O}(\phi)$. It is determined by the equation

$$\{\Omega_1, O\} = 0\,, \tag{4.21}$$

where Ω_1 is the leading term of expansion (3.2). In ref. [1] this path-integral representation for quantum averages was derived within a superfield approach to the BV-quantization of topological sigma-models.

5. Comparison with the field-antifield formalism

To gain further insight into the quantization of (non-)Lagrangian gauge theories described in the previous sections, it is instructive to compare our method with the standard field-anti-field formalism well-known for Lagrangian theories [2, 3]. We begin with a brief outline of the usual BV-formalism in the form which is suitable for comparing with its non-Lagrangian counterpart.

In the BV-formalism one starts with a configuration space $\mathcal{N}$ of fields ϕ^A endowed with an integration measure $D\phi$. The set ϕ^A includes both physical and ghost fields. The odd cotangent bundle $\Pi T^*\mathcal{N}$, with ϕ^*_A being fiber coordinates, is known as the field-anti-field supermanifold. Besides the Grassman parity, all the fields ϕ^A and anti-fields ϕ^*_A carry definite ghost numbers such that $\mathrm{gh}(\phi^*_A) = -\mathrm{gh}(\phi^A) - 1$. For simplicity sake, we assume that $\mathcal{N}$ is a super-domain with $D\phi$ being the canonical translation-invariant measure. The canonical measure on $\mathcal{N}$ induces the canonical measure $D\phi D\phi^*$ on the odd cotangent bundle $\Pi T^*\mathcal{N}$. Using these data, one can define the so-called *odd Laplace operator*[6]

$$\Delta = (-1)^{\epsilon_A}\frac{\partial_l^2}{\partial\phi^A\partial\phi^*_A}\,, \qquad \Delta^2 \equiv 0\,. \tag{5.1}$$

By definition, the operator Δ is nilpotent and has ghost number 1.

Let $W(\phi,\phi^*)$ be an arbitrary even function of ghost number zero. Define the *twisted* Laplace operator by the rule

$$\hat{\Omega} = e^{-\frac{i}{\hbar}W}(-\hbar^2\Delta)\,e^{\frac{i}{\hbar}W} = \hat{\Omega}_0 + \hat{\Omega}_1 + \hat{\Omega}_2\,, \qquad \hat{\Omega}^2 \equiv 0\,. \tag{5.2}$$

Here

$$\hat{\Omega}_0 = -i\hbar\Delta W + \frac{1}{2}(W,W)\,, \qquad \hat{\Omega}_1 = -i\hbar(W,\,\cdot\,)\,, \qquad \hat{\Omega}_2 = -\hbar^2\Delta\,, \tag{5.3}$$

and $(\cdot,\cdot)$ stands for the *antibracket* of two functions,

$$(-1)^{\epsilon(A)}(A,B) \equiv \Delta(AB) - (\Delta A)B - (-1)^{\epsilon(A)}A\Delta B\,. \tag{5.4}$$

Like Δ, the operator $\hat{\Omega}$ is odd, nilpotent, and has ghost number 1. The difference between $-\hbar^2\Delta$ and $\hat{\Omega}$ is in the first- and zeroth-order differential operators $\hat{\Omega}_1$ and $\hat{\Omega}_0$ constructed from W, see (5.3).

The operator $\hat{\Omega}$ is said to be the *quantum BRST operator* associated to the *quantum master action* W if

$$\Omega_0 = 0\,. \tag{5.5}$$

This condition is known as the *quantum master equation.* In the BV theory one is usually interested in *proper* solutions to the quantum master equation. These are specified by two additional requirements: (i) W is analytical in $\hbar$, i.e., $W = W_0 + \hbar W_1 + \hbar^2 W_2 + \cdots$, and (ii) the rank of the Hesse matrix d^2W_0 equals $\dim\mathcal{N}$ at any point of the stationary surface $dW_0 = 0$. In particular, the latter condition excludes the trivial solution $W = 0$ whenever $\dim\mathcal{N} \neq 0$.

[6] The BV method can also be formulated with a more general integration measure, see e.g. [19].

Thus, to define a gauge dynamics on $\Pi T^*\mathcal{N}$ means to specify a proper solution to eq. (5.5). The quantum BRST operator in the BV theory can be understood as a twist (5.3) of the odd Laplacian with W being the master action.

Given a quantum BRST operator, the physical observables are identified with zero ghost-number functions $O(\phi, \phi^*)$ obeying condition

$$\hat{\Omega} O = 0 \qquad \Leftrightarrow \qquad \Delta(O e^{\frac{i}{\hbar} W}) = 0 \,. \tag{5.6}$$

In view of (5.5) the unit $1 \in \mathbb{R}$ is automatically a physical observable. Of a particular interest is the following family of solutions to eq. (5.6):

$$\Phi_K = \delta(\phi^*_A - \partial_A K) e^{-\frac{i}{\hbar} W} \,, \qquad \hat{\Omega} \Phi_K = 0 \,, \tag{5.7}$$

$K(\phi)$ being an arbitrary odd function on $\mathcal{N}$ with ghost number -1. The function Φ^*_K is nothing but the gauge-fixed probability amplitude on the field-antifield configuration space $\Pi T^*\mathcal{N}$. The quantum average of a physical observables O is defined now as

$$\begin{aligned} \langle O \rangle = \int D\phi D\phi^* O \Phi^*_K = \int D\phi D\phi^* \delta(\phi^*_A - \partial_A K) e^{\frac{i}{\hbar} W} \\ = \int D\phi O(\phi, \partial K) \exp \frac{i}{\hbar} W(\phi, \partial K) \,. \end{aligned} \tag{5.8}$$

Because of (5.6) this integral does not actually depend on K (by Stokes theorem).

Notice that the BRST operator $\hat{\Omega}$ is hermitian w.r.t. the canonical integration measure providing the antifields ϕ^*_A are chosen to be pure imaginary. Using this fact and the definition (5.7) of the probability amplitude, one can easily deduce the Word identity

$$\langle O + \hat{\Omega} \Lambda \rangle = \langle O \rangle \quad \Leftrightarrow \quad \langle \hat{\Omega} \Lambda \rangle = 0 \,, \qquad \forall \Lambda \,. \tag{5.9}$$

In other words, the quantum average of a physical observable O depends on the $\hat{\Omega}$-cohomology class $[O]$ rather than a particular representative of this class.

The identification of rels. (5.2 -5.8) with analogous constructions of section 3 is now straightforward. The quantum BRST operator (5.2) is just a particular example of the general BRST charge defined by rels. (3.2, 3.9), (3.10) where $\hat{\Omega}_k = 0$, $\forall k > 2$, and the matrix Ω^{IJ} determining $\hat{\Omega}_2$ is constant and nondegenerate. The quantum master equation (5.5) corresponds to condition (3.11), the *flatness condition* in the terminology of ref. [1]. The physical values (5.6) are naturally identified with the BRST-invariant states (3.25). In particular, the value (5.7) - the complex conjugate to the gauge-fixed probability amplitude - coincides in form with a particular solution to the generalized Schwinger-Dyson equation (3.20), (3.41). Finally, the following relation set up a correspondence between minimal sectors of fields in a pure Lagrangian case[7]:

$$\phi^A = (\phi^i, c^\alpha) \,, \qquad \phi^*_A = (\eta_i, \xi_\alpha) \tag{5.10}$$

[7] As regards the non-minimal variables, one can introduce them in many different, but equivalent, ways [3].

(see Example 2 of section 3). With all these identifications, one can regard the BV path integral (5.8) as the quantum-mechanical matrix element (3.26), with the inner product being the ordinary L^2-norm w.r.t. the canonical integration measure on $\Pi T^*\mathcal{N}$.

As we have seen, the BV quantization scheme is a particular case of the proposed method that works also in the theories which are not Lagrangian. Now let us comment on the main properties of our method which may have a more general form in non-Lagrangian case in comparison with the Lagrangian BV scheme.

(i) In distinction from the canonical BRST operator (5.2), the general BRST charge (3.2) can involve differential operators of order $k > 2$ that gives rise to the *higher antibrackets* (see [20, 21], and references therein).

(ii) Even if all the higher antibrackets vanish (i.e., $\hat{\Omega}_k = 0, \forall k > 2$), the quantum BRST operators (5.2), (3.2) are not equivalent in general. This is because the antibracket (3.4) associated with the quadratic term in expansion (3.2) is allowed to be degenerate or even irregular as distinct from the canonical antibracket (5.4). This fact has further consequences:

(iii) In the BV quantization, the first-order operator $\hat{\Omega}_1 = (W, \cdot\,)$ is, by definition, an inner derivation of the canonical antibracket. For a degenerate antibracket (3.4) this is not always the case: the antibracket is still differentiated by the classical BRST differential $\hat{\Omega}_1$, but this may well be a non-inner derivation. In that case no quantum master action W can be found to twist the first order term $\hat{\Omega}_1$ out from the quantum BRST operator $\hat{\Omega} = \hat{\Omega}_1 + \hat{\Omega}_2$ by analogy with (5.2). This also means that the quantum BRST operator is not necessarily a twist (5.3) of the odd Laplacian $\hat{\Omega}_2$.

(iv) As a result, it turns out impossible to present the probability amplitude Φ^* of a non-Lagrangian theory as the exponential of some smooth function $e^{\frac{i}{\hbar}W}$ which is non-vanishing for every trajectory. The amplitude, being defined by the generalized Schwinger-Dyson equation (3.17), can be a more general distribution on the space of all histories. For example, the classical amplitude (1.13) vanishes everywhere except a classical solution.

As a final remark let us note that the proposed method, being applicable for covariant quantization of non-Lagrangian dynamics, can be also useful for in-depth study of gauge anomalies in conventional Lagrangian or Hamiltonian theories. Recall that in the BV-quantization, the anomalies manifest themselves as obstructions to solvability of the quantum master equation (5.5). In particular, the 1-loop anomaly is given by the modular class $[\Delta W_0]$ of the BRST-cohomology associated with the classical BRST-differential $\delta = (W_0, \cdot\,)$. Notice that the δ-cocycle ΔW_0 has ghost number 1 and is linear in the gauge algebra generators R as well as the higher structure functions (see (3.32)). Contrary to this, in the operator BFV-BRST quantization the same anomaly arises as violation of the nilpotency condition $\hat{\Omega}^2 = 0$ by quantum corrections which are obviously bilinear in the structure functions and have ghost number 2. So it might be not obvious in general how to relate these anomalies to each other. Moreover, in the operator quantization the very existence of anomalies depends crucially on the quantization scheme applied, not to mention their specific form. For instance, the 1-loop anomalies may occur in the Wick quantization but never in the Weyl one (see e.g. [9, 11]). What is a precise Lagrangian analogue for different symbols of the BRST operator is yet to be explored, but the proposed quantization scheme offers an alternative way to handle this problem. The point is that the gauge sector of the “Lagrangian BRST charge” (3.29) is quite similar to the Hamiltonian BRST charge constructed by the constraints R’s, with the only difference that the former is defined in the space of all histories while the latter corresponds to a fixed time moment. Both of these BRST operators obey the same master equation, $\hat{\Omega}^2 = 0$, leading to the same structure of quantum anomalies. The different symbols of operators on the genuine phase space of the system are then imitated by different symbols on the cotangent bundle of the space of all histories. In summary, the proposed quantization scheme combines the explicit covariance of the BV-quantization with the possibility to work with different symbols of the BRST operator, as is the case in the BFV-BRST-quantization. Besides, the proposed method extends these advantages beyond the reach of the BFV and BV schemes, making possible to explore quantum effects, including anomalies, in the theories admitting no Lagrangian formulation.

Acknowledgments

We are thankful to Jim Stasheff for his corrections to and comments on the first version of the manuscript. This work was partially supported by the RFBR grant 06-02-17352, Russian Ministry of Education grant 130337, and the grant for Support of Russian Scientific Schools 1743.2003.2. A.A.S. appreciates financial support from the Dynasty Foundation and International Center for Fundamental Physics in Moscow.

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JHEP02(2006)007